# Optimal Digital Flight Control for Advanced Fighter Aircraft

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**The operating characteristics of the next generation of fighter aircraft impose severe constraints which the control systems of these aircraft must overcome. The use of linear optimal digital control techniques allows one to answer some of the outstanding questions concerning how to design digital fly-by-wire control sytems. By using an "implicit model following" technique we have found that design specifications may be easily incorporated into the controller. By concentrating on noise and uncertainty we have found how to maximize the computer sample time, thereby reducing computer utilization. By appropriately incorporating all control surfaces we have found how to use these controls harmoniously to achieve the desired performance. Finally, all of these have been accomplished with a set of computer-aided design programs that are easily used and give rapid results. This paper describes the theoretical basis of our techniques and its application to a typical 1980's advanced fighter aircraft.**

## I. Introduction

IT is not the main point of this paper to present the theory of linear discrete-time optimal control. Therefore, we will briefly outline here the major assumptions made in designing optimal discrete systems. For further details the reader is referred to the excellent tutorial paper by Dorato and Levis.[1]

The continuous model of the aircraft is put into the state variable form (where all matrices are constant):

$$\dot{x} = Ax + Bu + Cw$$
$$y = Mx + v \tag{1}$$

where $x$ is an $n$ vector of aircraft states and any augmented noise states; $u$ is a $p$ vector control; $w$ is an $m_l$ vector noise process (which is assumed to be a white process); $y$ is an $m$ vector measurement; and $v$ is an $m$ vector white measurement noise. The discrete-time version of Eq. (1) is given by

$$x_{k+1} = \Phi(\Delta t)x_k + F(\Delta t)u_k + G(\Delta t)w_k$$
$$y_k = Mx_k + v_k \tag{2}$$

where

$\Delta T$ = the sample time;
$\Phi(\Delta t)$ = the transition matrix evaluated at $\Delta t$ (i.e., $e^{A\Delta t}$)
$F(\Delta t)$ = $\int_0^{\Delta t}\Phi(\Delta t-\tau)B\mathrm{d}\tau$ (i.e., $\int_0^{\Delta t}e^{A\tau}B\mathrm{d}\tau$);
$G(\Delta t)$ = a matrix which satisfies $G(\Delta t)G^T(\Delta t) = P(\Delta t)$, where $P$ is the covariance matrix of $x$ from Eq. (1) at $t=\Delta t$

In general, one would like to solve the following problem: "Given a quadratic performance index of the form

$$J = \int_{t_0}^{T}(x^TQx + u^TRu)\mathrm{d}t \tag{3}$$

find the control which minimizes $J$ subject to the constraint of the differential Eq. (1)."

The discrete version of Eq. (3) is obtained from Eq. (3) and the fact that the controls are held fixed over the sample interval $\Delta t$. In general, if one wants to compare the discrete design with the continuous design it is better to use a performance index for the discrete system which matches the continuous system. Thus, we use the performance measure

$$J = \sum_{k=1}^{N}(x_k^T\tilde{Q}x_k + 2u_k^T\tilde{S}x_k + u_k^T\tilde{R}u_k) \tag{4}$$

where

$$\tilde{Q} = \int_0^{\Delta t}\Phi^T(\Delta t-\tau)Q\Phi(\Delta t-\tau)\mathrm{d}\tau$$

$$\tilde{S} = \int_0^{\Delta t}F^T(\Delta t-\tau)Q\Phi(\Delta t-\tau)\mathrm{d}\tau\S$$

$$\tilde{R} = \int_0^{\Delta t}[R + F^T(\Delta t-\tau)QF(\Delta t-\tau)]\mathrm{d}\tau$$

The solution of the discrete-time control problem is obtained from Eqs. (2) and (4) by using either the discrete maximum principle or dynamic programming. In either case the ultimate solution is that $u_k$ is a linear function of the states of the system where the gain matrix $K_k$ satisfies

$$K_k = -(\tilde{R} + F^TP_kF)^{-1}(F^TP_k\Phi + \tilde{S}) \tag{5}$$

and $P_k$ is the solution of the discrete matrix Ricatti equation

$$P_k = (\Phi^TP_{k+1}\Phi + \tilde{Q}) - (F^TP_{k+1}\Phi + \tilde{S})^T$$
$$\cdot\ (\tilde{R} + F^TP_{k+1}F)^{-1}(F^TP_{k+1}\Phi + \tilde{S}) \tag{6}$$

with $P_N = [0]$.

If, as is generally assumed, the steady-state (constant) gain matrix is desired, we have found that the best way to solve for the gain, Eq. (5), is to use Potter's technique[2] of evaluating the eigenvalues and eigenvectors of the $2n\times 2n$ matrix which results from the application of the discrete maximum prin-

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§The matrix $F$ is defined in Eq. (2); also it is significant that the discrete performance index contains a nonzero matrix $\tilde{S}$ even though Eq. (3) does not.

ciple. Problems with extremely high order and with significant differences in time constants are as easily solved as low-order problems when using a numerical eigenvalue, eigenvector technique. If such problems were solved using iteration, the computer time required would be exorbitant. This matrix, for the performance index, Eq. (4), is given by $H$ where

$$H=\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right]$$

where

$$H_{11}=(\Phi-F\tilde{R}^{-1}\tilde{S})^{-1}$$

$$H_{12}=(\Phi-F\tilde{R}^{-1}\tilde{S})^{-1}F\tilde{R}^{-1}F^{T}$$

$$H_{21}=(\tilde{Q}-\tilde{S}^{T}\tilde{R}^{-1}\tilde{S})(\Phi-F\tilde{R}^{-1}\tilde{S})^{-1}$$

$$H_{22}=(\Phi^{T}-\tilde{S}^{T}\tilde{R}^{-1}F^{T}) + (\tilde{Q}-\tilde{S}^{T}\tilde{R}^{-1}\tilde{S})(\Phi-F\tilde{R}^{-1}\tilde{S})^{-1}F\tilde{R}^{-1}F^{T}$$

and it has the property that its eigenvalues are the $n$ poles of the closed-loop system and their reciprocals (there are $2n$ eigenvalues, $n$ stable and $n$ unstable). The steady-state gain is given by the Potter algorithm as follows:

Let $W$ be the matrix of eigenvectors, then

$$W^{-1}HW=\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix}$$

where $\Lambda$ represents the eigenvalues *outside* the unit circle. Then if $W$ is partitioned as

$$W=\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

the steady-state cost $P_{\infty}$ is given by

$$P_{\infty}=W_{21}W_{11}^{-1} \tag{7}$$

The program that is used to derive the optimal controls uses this technique to obtain the optimal gains. The $Q-R$ algorithm is used to determine the eigenvector matrix $W$. Potter's technique applies to both continuous and discrete systems; however, in the continuous version the eigenvalues of the $2n\times 2n$ matrix $H$ have symmetry about the imaginary axis (rather than radial symmetry). Since we use the continuous optimal design as a reference, the Potter solution for the continuous design existed first. To take advantage of this existing program we map the roots of the matrix $H$ using the bilinear transformation

$$w=(1-z)/(1+z)$$

This allows the roots inside and outside the unit circle to be selected based on their locations in the left-half or right-half $w$-plane.

The last point we have to consider is how one selects the sample time $\Delta t$. This has already been discussed in a former paper by the authors,[3] but a brief outline is in order here. The selection of a fundamental sample time uses the fact that between samples the system is essentially open-loop. Therefore, if perfect control at the sample times was achieved, between samples the uncertainty would propagate via the covariance matrix differential equations

$$\dot{P}=AP+PA^{T}+CC^{T}$$

$$P(0)=[0] \tag{8}$$

At the first time $t$ that the uncertainty due to this open-loop propagation exceeds a bound specified by the control specifications, there *must* be a sample to lower this uncertainty. Since we are neglecting uncertainties that are due to the control (feedback) and the state estimation in doing the propagation of Eq. (8) (i.e., $P(0)$ is really not $[0]$), the actual sample rate should be modified based on the closed-loop noise response. As we will see in Sec. III, the existence of an automated design program allows this to be easily implemented.

## II. Implicit Modeling Following

The theory described in the introduction has been faulted in the past for impracticality because one does not have a simple way of selecting $Q$ and $R$ in the performance index, Eq. (3). Furthermore, the theory presented is valid for the class of problems which are known as "regulator" control, where the ultimate desire is to maintain the state at the point $\boldsymbol{x}=\boldsymbol{0}$. Aircraft control, in distinction, belongs to the class known as "servo" problems where the desire is to track the pilot commands.[4] The two disparate problems of performance index selection and that of control for input tracking may be solved using a technique known as "model following."

The technique that we have found most successful in creating designs which meet a specified performance criterion is the "implicit model following" or "model in the performance index" technique.[5-7] This design approach commences with the desired model, which is a differential equation that describes the specified response of the aircraft. Thus, if $\boldsymbol{x}_m$ is the model state, it is assumed that

$$\dot{\boldsymbol{x}}_m=A_m\boldsymbol{x}_m+B_m\delta_c \tag{9}$$

where $\delta_c$ is the command input. The command input is further modeled as a differential equation (a pilot model) of the form

$$\dot{\boldsymbol{\delta}}_c=A_c\boldsymbol{\delta}_c \tag{10}$$

with

$$\delta_c(0)=\delta_{c_0}$$

Both of these equations are combined to produce an augmented model state and also an augmented aircraft state by combining Eqs. (10) with (1). Since we are interested in the discrete-time model, the equations are converted to discrete time. Thus, Eqs. (1) and (10) together produce the state $\boldsymbol{x}_a$, defined as follows:

$$\dot{\boldsymbol{x}}_a=\begin{bmatrix} \dot{x} \\ \dot{\delta}_c \end{bmatrix}=\begin{bmatrix} A & 0 \\ 0 & A_c \end{bmatrix}\begin{bmatrix} x \\ \delta_c \end{bmatrix}+\begin{bmatrix} B \\ 0 \end{bmatrix}u$$

which leads to a discrete time model

$$\boldsymbol{x}_{a_{k+1}}=\begin{bmatrix} \Phi & 0 \\ 0 & \Phi_c \end{bmatrix}\boldsymbol{x}_{a_k}+\begin{bmatrix} F \\ 0 \end{bmatrix}u_k \tag{11}$$

Also, Eqs. (10) and (9) provide the augmented state $z$ given by

$$\dot{z}=\begin{bmatrix} \dot{x}_m \\ \dot{\delta}_c \end{bmatrix}=\begin{bmatrix} A_m & B_m \\ 0 & A_c \end{bmatrix}\begin{bmatrix} x_m \\ \delta_c \end{bmatrix}=A_M z$$

or, in discrete time

$$z_{k+1}=\Phi_M z_k$$

where

$$\Phi_M=e^{A_M\Delta t} \tag{12}$$

The output of the actual system now consists of the original measurement $y$ augmented by the states of the command generator $\delta_c$. Therefore, the linear optimal solution derived here will be feedback gains on the state $x$ (through the measurement $y = Mx$) and feedforward gains from the pilot controller (through the gains on $\delta_c$). Our ultimate desire is to have the aircraft match the model. Therefore, if we define the augmented measurement $\tilde{y}_k$ as

$$\tilde{y}_k = \begin{bmatrix} Mx_k \\ \delta_{c_k} \end{bmatrix}$$

and if the states in the model which are to be matched by $\tilde{y}$ are given by $y'_k = Cz_k$ ($y'$ is the same dimension as $\tilde{y}$), then the model match in these states is achieved by modifying the performance index, Eq. (4), as follows

$$J_i = \sum_{k=1}^{\infty} (\tilde{y}_{k+1} - y'_{k+1})^T \tilde{Q} (\tilde{y}_{k+1} - y'_{k+1}) + 2u_k^t \tilde{S} (\tilde{y}_{k+1} - y'_{k+1}) + u_k^T \tilde{R} u_k \tag{13}$$

The optimum control will now attempt to reduce $\tilde{y}_k$ to $y'_k$ which, if it can be achieved, will make the actual aircraft match the model. (Note that the presence of $\delta_c$ in both $\tilde{y}$ and $\tilde{y}'$ means that nothing is contributed to $J_i$ by the command state $\delta_c$.) Combining Eqs. (11) and (12) with Eq. (13) results in the modified performance index

$$J = \sum_{k=1}^{\infty} (x_k^T \hat{Q} x_k + 2u_k^T \hat{S} x_k + u_k^T \hat{R} u_k)$$

where

$$\hat{Q} = (M\Phi - \Phi_M M)\tilde{Q}(M\Phi - \Phi_M C) \tag{14a}$$

$$\hat{S} = \tilde{S} + F^T M^T \tilde{Q}(M\Phi - \Phi_M M) \tag{14b}$$

$$\hat{R} = \tilde{R} + F^T M^T \tilde{Q} M F \tag{14c}$$

The performance index, Eq. (14), is now the starting point for the optimal design as outlined in the introduction. Several points can now be made. First, even if a very simple $Q$ and $R$ were assumed, the $\hat{Q}$, $\hat{S}$, and $\hat{R}$ that result is a performance index quite different and one which no amount of judgment would lead the designer to select, *a priori*, as his measure of "goodness." Second, there is no explicit model in the control loop, and therefore, initial condition mismatch between model and actual aircraft states does not occur. Third, explicit incorporation of the command into the dynamics has made it possible to do the servo problem using the optimal regulator approach.

## III. Advanced Fighter Aircraft – Problem Description

We have applied the theory described above to an advanced fighter aircraft with several unique operating characteristics. First, the vehicle is statically unstable in some of its flight conditions. Second, the assumption was made that an inboard trailing-edge flap was available for lift-pitch control as well as a conventional elevator. We have selected a flight condition that permits very high accelerations to be commanded, and one which has an unstable dynamic description (Mach 0.6, 15,000 feet altitude). The design we will show is for the longitudinal control system to achieve an 8-$g$ pull-up command from level flight (1$g$) in less than 0.8 second with an overshoot (from structural considerations) of no more than 0.5 $g$. The model that would achieve the specifications has the state variable form

$$\dot{x}_m = \begin{bmatrix} -0.046 & 0.0085 & -8.46 & -32.2 \\ -0.124 & -2.05 & 633 & 0 \\ 0.003 & -0.05 & -7.6 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_m + \begin{bmatrix} 0 & 0 \\ -8.5 & 0 \\ 1.8 & 0 \\ 0 & 0 \end{bmatrix} \delta_c \tag{15}$$

where the state $x_m$ consists of longitudinal velocity, normal velocity, pitch rate, and pitch angle.

The command is to be a step which has an exponential lag and is modeled as

$$\dot{\delta}_c = \begin{bmatrix} 0 & 1 \\ 0 & -6.66 \end{bmatrix} \delta_c \text{ with } \delta_c(0) = \begin{bmatrix} 0 \\ 2.6 \end{bmatrix} \tag{16}$$

Equation (16) implies that both stick position and stick rate are measured, and the initial $\delta_c$ implies that when the pilot hits the stick his hand is already moving. The measurement of stick rate is an important aspect of this design since it shortens the delay between stick motion and aircraft response, particularly when sample times are long.

The flight condition we are modeling is one which has unstable dynamics. The longitudinal equations of motion are given by (the state vector consists of longitudinal velocity, normal velocity, pitch rate, pitch angle, and actuator angle)

$$\dot{x} = \begin{bmatrix} -0.046 & 0.0086 & -8.46 & -32.2 & 0 \\ -0.1124 & -2.05 & +633 & 0 & -230.8 \\ +0.0003 & +0.04 & -1.518 & 0 & -21.5 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20 \end{bmatrix} u \tag{17}$$

where $u$ is the input to the stabilizer actuator. The open-loop $z$-plane poles corresponding to the sampled version of Eq. (17), at a sample rate of 10 sec$^{-1}$, are at 0.135, 0.5057, 1.384, 0.9976 $\pm i0.0075$, 0.6938 $\pm$ $i0.2287$. The pole at 1.384 is the instability.

## IV. Optimal Design – Sample-Time Turbulence Response and Transient Response

To derive the best sample time for the vehicle described in Sec. III, a turbulence that matches the "Dryden" spectrum[8] was used as the predominant system noise. The dynamics of this spectrum have been formulated as a second-order state variable model excited by white noise as follows:

$$\dot{x}_n = \begin{bmatrix} 0 & 1 \\ -0.131 & -0.725 \end{bmatrix} x_n + \begin{bmatrix} 5.21 \\ -2.69 \end{bmatrix} w(t) \tag{18}$$

The open-loop covariance matrix was propagated by combining Eqs. (17) and (18) into an augmented noise model of the form of Eq. (1). The uncertainties we can tolerate were specified as $3\sigma$ bounds. The elements along the diagonal of the covariance matrix were compared with these bounds to determine the largest time between samples which will allow the state uncertainties to be within these bounds. Since this technique ignores control uncertainties, the closed-loop system turbulence sensitivities were derived by iterating on the closed-loop design. First, the optimal continuous design (sample time equal to zero) was derived, and the variance of the controlled states with the noises, Eq. (18), was computed. Second, the sample time was increased and, once again, the state variances were computed. Figure 1 is a plot of the worst uncertainty, that of the state $n_z$ (normal acceleration), as a function of sample time. The normal acceleration bound that was considered acceptable is illustrated in the figure and, as can be seen, the sample time implied is 0.1 sec. (Once again it should be emphasized that these are closed-loop "noise" responses.)

In a prior paper,[3] this sample-time approach was used to design a system with bending modes. It was shown there that the use of an optimal filter can result in sample times which are on the order of 0.1 sec, even though there existed a 20-Hz bending frequency.

A computer program, which we call DIGISYN, was used to derive all of the results described in this paper. Figure 2 shows the modeled normal acceleration and the resulting response obtained from DIGISYN using the 0.1-sec sample time. The result of additional controls can also be seen in Fig. 2 where an inboard flap was included in Eq. (17) as an additional control. The inboard flap makes several important improvements possible. First, the acceleration reversal that is felt at the c.g. of the airplane when only the stabilizer control is used is eliminated. Second, the gust performance is improved dramatically, as can be seen in Fig. 1. Thus the optimal design, with the flap control, is a gust alleviation system designed as a by-product of optimizing the control configuration.

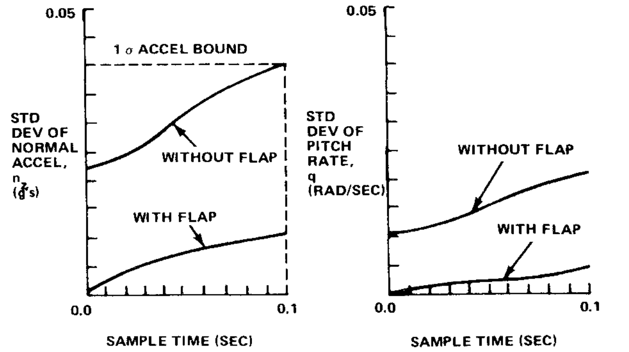


**Fig. 1 Gust performance as function of sampling rate (5 fps gust).**

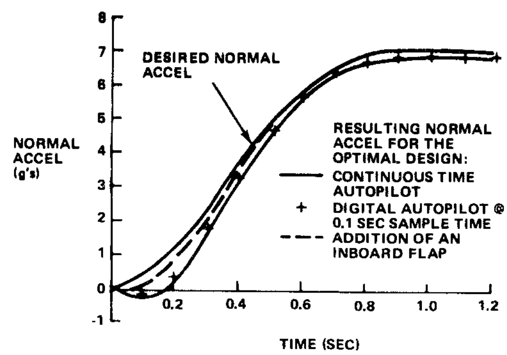


**Fig. 2 Optimal design for step input.**

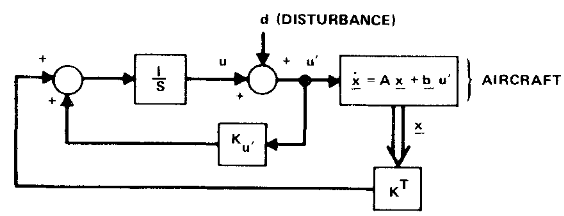


**Fig. 3 Full-state feedback control with integral forward loop compensation.**

## V. Optimal Design – Integral Compensation and Different Specifications

The aerodynamic characteristics of the vehicle we are considering exhibit a "speed instability" that requires an elevator rate reversal to maintain trim during level flight acceleration. This characteristic requires some form of automatic trim. Classic control theory concepts call for the design of a Type I system to handle the nonzero steady-state error, i.e., integral compensation is required. Figure 3 shows a block diagram of an optimal design where integral compensation in the forward loop is implied. The disturbance term, $d$, represents the elevator trim requirement so that $u'$ is the "perturbation" elevator command. If the aircraft equations of motion, Eq. (17), are augmented by an integrator equation, then the implicit model-following method described above may be readily used to obtain all of the gains ($K_{u'}$ and $K^T$) in Fig. 3. To simplify the material presented here, we have eliminated the phugoid mode from Eq. (17) and consider only the augmented pitch plane dynamics described as follows

The feedback gains $K_I$ and $K_P$ for the discrete-time model will be slightly different from those derived by Anderson; they

$$\dot{x} = \begin{bmatrix} \dot{w} \\ \dot{q} \\ \dot{\delta}_e \\ \dot{u}' \\ \dot{\delta}_c \\ \ddot{\delta}_c \end{bmatrix} = \begin{bmatrix} -2.46 & 634.4 & -277.2 & 0 & 0 & 0 \\ 0.066 & -1.271 & -36.05 & 0 & 0 & 0 \\ 0 & 0 & -20 & 20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -6.6 \end{bmatrix} \begin{bmatrix} w \\ q \\ \delta_e \\ u' \\ \delta_c \\ \dot{\delta}_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{u} \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

where $u' = u + d$ and $\dot{u}' = \dot{u}$; $w,q$ are the aircraft states; $\delta_e$ is elevator angle; and $\delta_c$, $\dot{\delta}_c$ are the command generator states.

The implicit model-following parts of DIGISYN give an optimal gain for each state variable in Eq. (19). However, if one reflects for a moment on the block diagram of Fig. 3, it can be seen that the state $u'$ cannot be measured. Anderson and Moor[4] suggest a block diagram manipulation which results in the system of Fig. 4.

are related to the gains in Fig. 3 by the pseudoinverse of the discrete control influence matrix $f$ [the discrete form of the vector multiplying $\dot{u}$ in Eq. (19)]

$$K_P^T = K_{u'}(f^T f)^{-1} f^T \tag{20a}$$

$$K_I^T = [K^T - K_{u'}(f^T f)^{-1} f^T \Phi] \tag{20b}$$

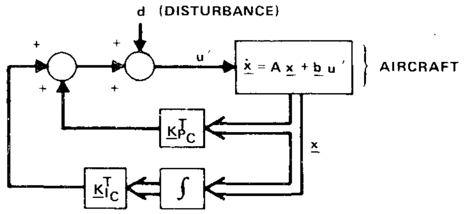


Fig. 4 Alternate integral compensation form ($K_{IC}$, $K_{PC}$ are continuous time gains).

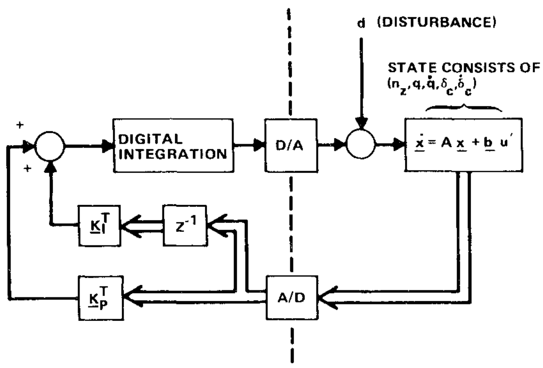


Fig. 5 Digital control system form of alternate compensation.

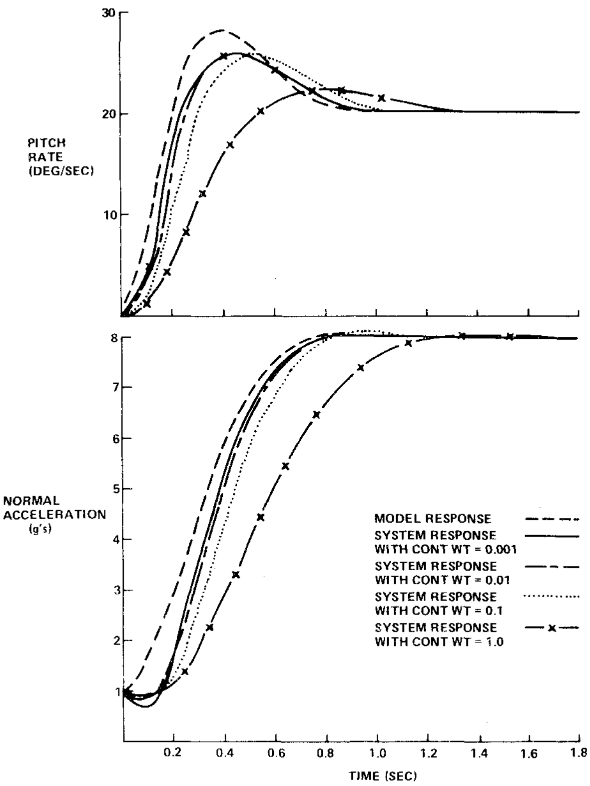


Fig. 6 Step response of system with varying control weights.

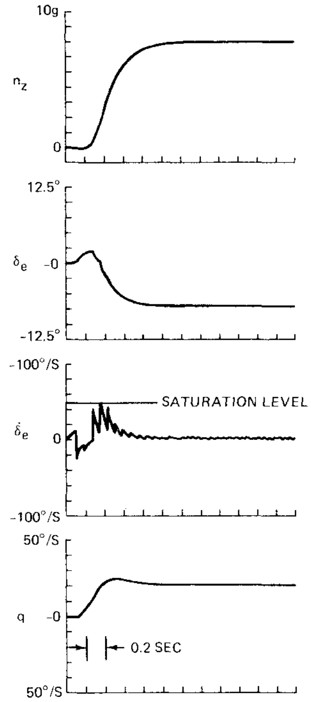


Fig. 7 Maximum $g$ maneuver, 50°/sec actuator rate limit.

The actual implementation of the feedback system will use measurements of normal acceleration ($n_z$), pitch rate ($q$), and pitch acceleration ($\dot{q}$). This means that the states of Eq. (19) are not those which we would like to have the feedback gains multiply. Since the states $n_z$, $q$, and $\dot{q}$ are linear combinations of the states in Eq. (19), we have the new state vector $x' = Tx$, where $x$ is the old state. Therefore, instead of Eq. (19) we have the equations of motion

$$\dot{x}' = T^{-1}ATx' + T^{-1}bu' \tag{21}$$

or the gains which we derived for the model Eq. (21) are now given by $K_{\text{new}} = K_{\text{old}}T^{-1}$ where $K_{\text{old}}$ is the gain matrix derived for Eq. (19). Figure 5 illustrates the feedback system with integral compensation as it would be implemented.

The results of applying DIGISYN to the integral compensation model described above are summarized by the gains in Table 1. The control weight in column one of the table is the only parameter appearing in the performance index which is selected by the designer. That is, in the performance index of Eq. (3) the $Q$ matrix is the identity and the $R$ matrix is $\alpha$ times the identity. The quantity $\alpha$ acts like a LaGrange multiplier and is used to guarantee that the actuator maximum rate constraints are not exceeded by the worst-case command for which we are designing the control system. Figure 6 shows the digital controller responses in pitch rate and normal acceleration corresponding to weights $\alpha$ from 0.001 to 1. The control weight of 0.1 was verified to be that which gives the best compromise between actuator rate saturation constraints and attainment of the performance specification. Figure 7 shows response characteristics from an analog computer simulation (with nonlinear actuator dynamics) using gains for $\alpha = 0.1$. As can be seen, the worst-case maneuver does not cause significant actuator rate saturation, and the response is quite adequate. One last point should be made concerning the gains in Table 1. In an absolute sense, none of these gains is large. This is quite different from what would occur if an explicit model control

Table 1 Control gains

| | | | Gains to integrator | | | Proportional gains | | |
|---|---|---|---|---|---|---|---|---|
| Control weight ($\alpha$) | $K_{\delta_c}$ | $K_{\dot{\delta}_c}$ | $K_{q_I}$ | $K_{n_{z_T}}$ | $K_{\dot{q}_I}$ | $K_{q_P}$ | $K_{n_{z_P}}$ | $K_{\dot{q}_P}$ |
| $10^{-3}$ | −0.545 | −0.04 | 5.03 | 0.018 | 0.68 | 0.024 | −0.016 | 0.019 |
| $10^{-2}$ | −0.43 | −0.035 | 4.35 | −0.0056 | 0.59 | 0.022 | −0.015 | 0.017 |
| 0.1 | −0.236 | −0.023 | 3.15 | −0.042 | 0.44 | 0.018 | −0.012 | 0.014 |

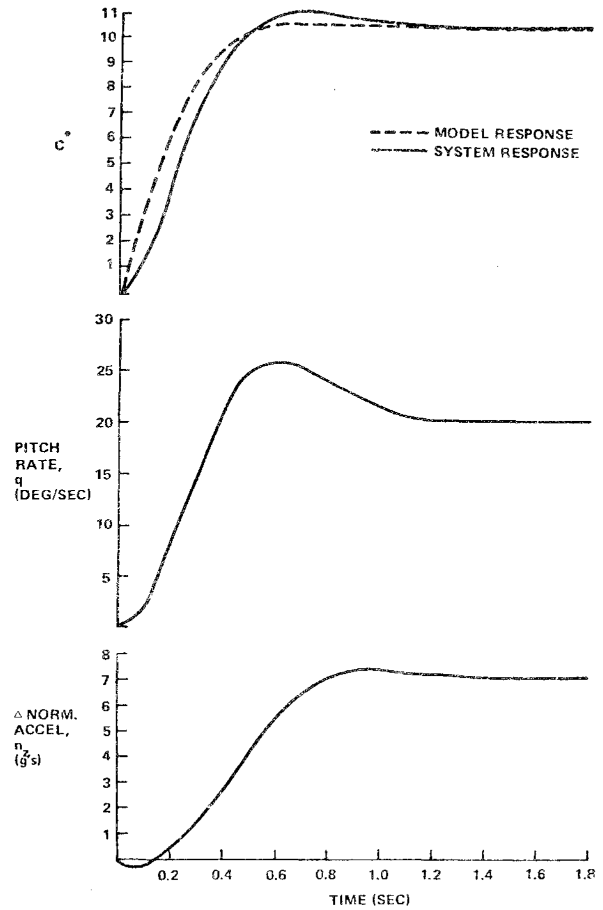


**Fig. 8 Step response using a $C^*$ model.**

were derived. The actual incorporation of a model in the loop would cause very high feedback gains (relative to those we have derived) to be used.

The final study we made using the high-performance aircraft described here was to explore different specifications. A popular, albeit controversial, specification is the "$C^*$ spec," where $C^*$ is a linear combination of the states $n_z$, $q$, and $\dot{q}$ given by[9]

$$C^* = K_1 n_z + K_2 q + K_3 \dot{q} \tag{22}$$

where $K_1$, $K_2$, and $K_3$ are functions of the flight conditions and aircraft geometry. In general, a transfer function for $C^*$ to $\delta_c$ is given by

$$\frac{C^*(s)}{\delta_c} = \frac{b_2 S^2 + b_1 S + b_0}{S^2 + a_1 S + a_0} \tag{23}$$

which may readily be put into a state variable form augmented by the command generator as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{\delta}_c \\ \ddot{\delta}_c \end{bmatrix} = \begin{bmatrix} 0 & 1 & b_1 - b_2 a_1 & 0 \\ -a_0 & -a_1 & b_0 - b_2 a_0 - c_1 a_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -6.67 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \delta_c \\ \dot{\delta}_c \end{bmatrix} \tag{24}$$

where

$$c_1 = b_1 - b_2 a_1 \tag{24a}$$

$$z_1 = C^*(t) - b_2 \delta_c(t) \tag{24b}$$

$$z_2 = \dot{z}_1 - c_1 \delta_c(t) \tag{24c}$$

Since there is now a difference between the model states and those defined by the system state variables in Eq. (19), a new measurement matrix $M$ must be defined. Thus, since the states in Eq. (24) are linear combinations of those in Eq. (19) we have

$$\begin{bmatrix} z_1 \\ z_2 \\ \delta_c \\ \dot{\delta}_c \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 & M_{15} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega \\ q \\ \delta_e \\ u' \\ \delta_c \\ \dot{\delta}_c \end{bmatrix} \tag{25}$$

where the $M_{ij}$ are functions of the coefficients of the matrix $A$ in Eq. (1) and the gains $K_1$, $K_2$, and $K_3$ in the definition of $C^*$. The results of using DIGISYN to design a $C^*$-following controller is shown in Fig. 8.

One point concerning the $C^*$ model-following is important. Since the differential equation for $C^*$ contains $\delta_c$, when the model, Eq. (24), is included in the performance index, the fact that $\delta_c$ is nonzero in steady state and that the $\hat{Q}$ we have is positive definite means that $J_i$ will go to infinity as the terminal time goes to infinity (the steady-state cost is infinite). This will manifest itself in the computation of the gains using Potter's method in the singularity of the matrix $W_{11}$ in Eq. (7). We have found two different techniques for circumventing this problem. The first relies on the fact that the $Q-R$ algorithm which generates $W$ in Eq. (7) gives good answers for most of $W$, and the inverse is computable because of numerical errors which do not make $W_{11}$ exactly singular. The result is that $W_{21} W_{11}^{-1}$ in Eq. (7) is accurate everwhere but in the last columns corresponding to the command generator states. Since $W_{21} W_{11}^{-1}$ should be symmetric, by simply using the last rows as the last columns the gains are accurately computed. Our second approach to this problem extracts the command generator states from the matrix $H$ and treats them separately.

## VI. Conclusions

The design of an optimal digital flight control system using modern sampled data control theory can lead to a complete design in a very short time. Furthermore, the sophistication of this design is such that many different independent control functions, from gust alleviation to static instability enhancement, may be easily incorporated as part of a design which simultaneously satisfies the control specifications.

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